Experimental Investigation of Modal Power Distribution in a Duct at High Frequency

S. M. Baxter* and C. L. Morfey†
The University of Southampton, Southampton, England

Sound attenuation in a lined duct depends on the distribution of acoustic power among the propagating modes. This paper describes the application of a new data analysis technique for investigating in-duct modal power distributions. The technique is designed for high frequencies, with many more propagating modes present than available data points, so that mode-by-mode analysis techniques are impractical. The technique involves fitting continuous parametric models of the modal distribution to experimental data. Using data from a twelve-point microphone traverse experiment, information has been obtained about the structure of a duct field at frequencies high enough to allow as many as 500 modes to propagate.

Nomenclature

c_0	= sound speed
C	= normalized complex cross-spectral density
$C_{\mathrm{obs}}, C_{\mathrm{pred}}$	= observed and predicted values of C, respectively
I	= plane-wave weighting function
I_p^q	= spherical harmonic coefficient
$\boldsymbol{k}$	= vector wave number
k	= $\lvert \boldsymbol{k} \rvert$
k_{ax}	= axial component of wave number
M	= number of sampling positions
N	= number of spherical harmonics
$\boldsymbol{r}$	= $\boldsymbol{x}-\boldsymbol{x}'$
S_N	= model composed of N harmonics
u,v	= azimuthal and polar spherical angles, respectively, referred to duct axis
W	= weighting factor
$\boldsymbol{x},\boldsymbol{x}'$	= position vectors
Y_p^q	= spherical harmonic function
ω	= radian frequency

Introduction

SOUND attenuation in a lined duct depends on, among other factors, the distribution of acoustic power among the propagating modes. Conclusions drawn about the properties of the liner from measurements of overall sound attenuation therefore depend on a knowledge of, or assumptions about, this distribution. However, mode-by-mode investigations of the distribution become impractical at high frequencies when many ($N>20$) propagating modes are present.

The purpose of the project outlined in this paper has been to develop and apply a new data analysis technique for investigating a duct's modal power distribution. The technique is intended to work at high frequencies, with many more propagating modes present than available data points. A smoothed version of the modal power distribution is found by fitting parametric models to a limited number of two-point microphone measurements inside the duct.

Received Feb. 11, 1983; presented as Paper 83-0731 at the AIAA Eighth Aeroacoustics Conference, Atlanta, Ga., April 11-13, 1983; revision received Dec. 16, 1983.

*Research Student, Institute of Sound and Vibration Research.

†Reader in Fluid Dynamics and Acoustics, Institute of Sound and Vibration Research. Member AIAA.

The new analysis technique, together with theoretical results on which it is based, has been presented earlier[1,2] and is summarized below. The technique now has been fully developed and the main purpose of the present paper is to describe its application to a set of experimental data.

Description of Proposed Analysis Technique

Summary of Theoretical Approach: Free-Wave Fields

The new modal analysis technique is based on a description of the sound field as a "free-wave" field. A free-wave field is made up of uncorrelated plane waves with angle-dependent energy.

The basic data quantity used in the analysis has been C, a normalized complex cross-spectral density (CCSD), taken between the signals from a pair of microphones in the duct field. If the microphones are at positions $\boldsymbol{x},\boldsymbol{x}'$, then the quantity C is defined by

$$C(\omega,\boldsymbol{x},\boldsymbol{x}') = \mathrm{CCSD}(\omega,\boldsymbol{x},\boldsymbol{x}')/\{\mathrm{PSD}(\omega,\boldsymbol{x})\mathrm{PSD}(\omega,\boldsymbol{x}')\}^{1/2} \tag{1}$$

CCSD denotes the signals' CCSD, PSD denotes each signal's power spectral density, and ω is radian frequency.

If the field is free-wave, second-order statistics of the field like C are spatially homogeneous, and can be expressed in the form

$$C(\omega,\boldsymbol{r}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}u \int_0^{\pi} \mathrm{d}v \, \sin v \; I(\omega,v,u)\exp(i\boldsymbol{k}\cdot\boldsymbol{r}) \tag{2}$$

where $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{x}'$. The integral is a weighted sum of uncorrelated plane-wave terms. Each wave is specified by azimuthal and polar spherical angles u,v; the wave's wave number vector has polar coordinates (k,v,u) with $k=\omega/c_0$. [Note that the function I in Eq. (2) is equivalent to the function $\hat{I}$ in Eq. (23), Ref. 1.]

References 1 and 2 show that in a hard-walled rectangular or circular duct at high frequencies (i.e., the duct radius is large compared to the wavelength) the multimode sound field can approach a free-wave form. A necessary condition for this to hold is that the modes be uncorrelated; another, in the case of a circular duct, is that the modal power weighting should depend only on axial wave number.

Furthermore, the plane-wave weighting function I of Eq. (2) can be interpreted in terms of the modal power distribution as follows. Let the axis of the duct be parallel to the z axis. Associated with a mode at a given frequency is a polar angle v defined by

$$k_{ax} = k\cos v \qquad (3)$$

where k_{ax} is the mode's axial wave number. There are several modes associated with a neighboring small range of angle dv. Suppose, at a given frequency, the modal power weighting is a function of axial wave number alone. Then the function I is dependent only on polar angle v, and the value of $I(\omega,v)$ for a given v is proportional to the power carried by *each of the modes* associated with a neighboring small range of angle dv. (See the Appendix for an outline proof of this result. Detailed proof for the rectangular hard-walled duct, with no assumption of axial symmetry, is given in Sec. II.B of Ref. 1.) I is normalized so that, at every frequency,

$$\int_0^\pi dv \sin v \, I(\omega,v) = 1 \qquad (4)$$

is satisfied.

For example, if all of the power is distributed equally among the modes propagating in the positive z direction, then the function I will be zero for $\cos v < 0$, and one for $\cos v > 0$. If the duct has a perfectly reflecting termination, so the power is distributed equally among the modes propagating in both directions, then the function I has the value ½ for every value of $\cos v$ (see Fig. 1).

The assumptions behind this description are that the frequency is high enough, and the modal power distribution smooth enough, for the discrete modal distribution to be approximated by a continuous function of the angle v.

Parametric Model Fitting

The proposed data analysis technique begins from the assumption that the given duct sound field is of free-wave form. Experimental measurements of the quantity C yield information about the weighting function I via Eq. (2) and, hence, information about the modal power distribution.

The technique involves families of parametric models of the function I. Substitution of a given model I into Eq. (2) yields a prediction $C_{pred}(\omega,r)$ for $C(\omega,r)$ at any given position and frequency. The prediction depends on the values of the model's adjustable parameters.

At a given frequency ω, let there be available a set of observations of C $\{C_{obs}(\omega,r_j): j=1,M\}$ taken at M different microphone separations r_j. The values of the model's parameters can be chosen by performing a least-squares fit to the data, with the following residual sum of squares (rss) being minimized:

$$\text{rss} = \sum_{j=1}^{M} |C_{obs}(\omega,r_j) - C_{pred}(\omega,r_j)|^2 W(r_j) \qquad (5)$$

[$W(r_j)$ is a weighting factor.] Information about I as a function of angle and frequency then can be gleaned from the parameter values.

A least-squares fitting technique was chosen because of its robustness with regard to the experimental geometry. The present experiment was a microphone traverse involving a sparse irregular array of sampling positions.

Spherical Harmonic Functions

The main family of model I functions considered was a set of combinations of N spherical harmonic functions

$$S_N(\omega,v,u) = \sum_p^N \sum_q I_p^q(\omega)\, Y_p^q(v,u) \qquad (6)$$

The coefficients I_p^q are the model's adjustable parameters. Each Y_p^q is a combination of trigonometric functions of v (with maximum order p) and of u (with maximum order q).

Spherical harmonics are a complete, orthonormal set of functions of angle. That is, for a given function of angle, there is for each N a sum S_N of the form of Eq. (6) which is the best Nth-order spherical harmonic approximation to that function. (The approximation is "best" in a least-squares sense.) If the function is known exactly, the coefficients in the sum may be determined exactly. The convergence of S_N to the target function improves as N increases.

Examples of low-order approximations to a known function are shown in Fig. 2. Spherical harmonic approximations have the drawback that they can take negative values even if the target function is always positive, as illustrated. The plane-wave weighting function I is always positive, but harmonic approximations to it may not be.

The spherical harmonic coefficient values yielded by the experimental data analysis technique described above are estimates of the true coefficient values. As the number of data points M increases, the accuracy of the estimated coefficients is improved, and the number of terms N in a possible fitting series increases. The coefficient estimates are also affected by statistical errors in the experimental data.

Data Analysis Programs

Search computer programs were written to find well-fitting combinations of harmonics. Each search was a forward-searching iterative procedure involving a "hopper" of M low-order harmonics. When a model of N harmonics had been found, an improved fit to the data was sought by adding to the model the harmonic still in the hopper which fitted best to the residual data. Harmonics could also be deleted from the model and returned to the hopper if their contribution to the overall fit fell below a threshold of significance. The search was terminated by statistical significance tests. Each search yielded several models of varying degrees of significance.

As a safeguard against spatial aliasing, no combinations were accepted with high correlations between pairs of harmonics. The analysis technique was validated by applying it to sets of synthetic data with known spherical harmonic coefficients.

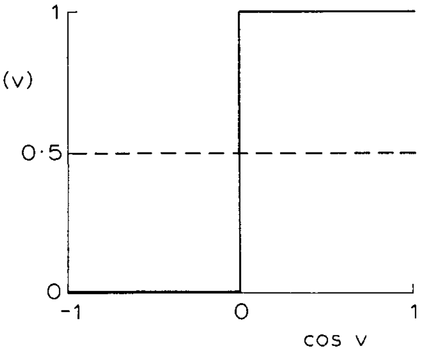

Fig. 1 Theoretical plane-wave weighting functions: ——, power equally distributed among modes traveling in one direction; ---, power equally distributed among modes traveling in both directions.

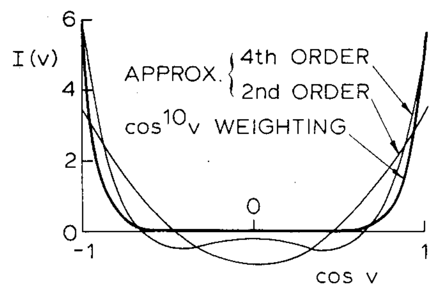

Fig. 2 Low-order harmonic approximations to a known plane-wave weighting function. Known weighting is proportional to $\cos^{10} v$.

Application of Proposed Technique to Experimental Data

Description of Experiment

The parametric model fitting technique described previously was applied to data from an experiment performed in the Noise Test Facility (NTF) of the National Gas Turbine Establishment, Farnborough, U.K. The NTF is a large-scale duct absorber test facility. The duct was hard walled and circular (radius 34.29 cm) with no flow present, and the broadband sound field in the duct was excited by four Hartmann generators in a reverberation chamber. The reverberation chamber was connected to one duct termination by a tapered inlet. At the other termination was a similar reverberation chamber. A twelve-point microphone traverse was performed. The frequency range 2-5 kHz (100-500 propagating modes) was studied.

A fuller description of the experiment, together with a preliminary analysis of the data, has been given earlier.[1]

The normalized CCSD C obtained at a typical sampling position is shown as a function of frequency in Fig. 3. The data analysis technique described above was applied to sets of C values taken at 250 Hz intervals over the range 2-5 kHz. The C value predictions from the fitting procedures are indicated in Fig. 3.

Description of Results

Typical plane-wave weighting function models derived by using the new data analysis technique are shown in Fig. 4. The data used to produce these models was the 5 kHz set, and a "hopper" consisting of low-order axisymmetric harmonics was used (that is, the models were constrained to be independent of azimuthal angle u). The order of the harmonic approximations in Fig. 4 is similar to that used in Fig. 2 to fit $\cos^{10} v$, and the similarity of the results should be noted.

Models found by the search programs for a given data set were typically found to be grouped into sets of models with comparable harmonic coefficients and statistical significance; examples for the 5 kHz data set are given in Fig. 5. A final model may be chosen nominally, with an uncertainty represented by the set of comparable models from which it was taken.

At all frequencies (2-5 kHz) in the present data set, the weighting function appeared to be independent of azimuthal angle u (so that the underlying modal power distribution is dependent only on axial wave number). This was established by comparing the results of the axisymmetric analysis with the results from a more general analysis including nonaxisymmetric terms.[3] No significant differences were found.

In Fig. 6, second-order models of $I(v)$ from each of the 13 frequencies are overlaid. At this level of resolution, no trend with frequency is apparent.

The stability of the models to statistical errors in the data was tested by applying the search programs to data sets with different variances (that is, different numbers of degrees of freedom), obtained by analyzing microphone signals of different duration, with the bandwidth kept constant. The effect of data error on a typical model set is illustrated in Fig. 7. The coefficients are stable for reasonably small variances; an ensemble size of 500 degrees of freedom appears to be necessary.

Discussion of Results

The similarity between Figs. 2 and 4 appears to indicate that the plane-wave weighting function for 2-5 kHz sound in the

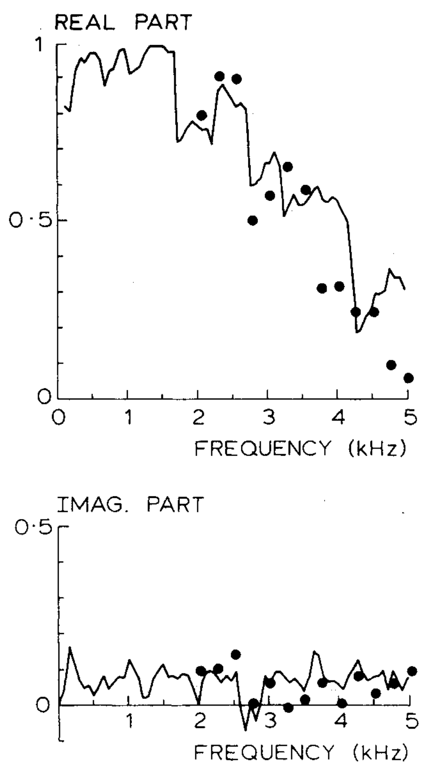

Fig. 3 Normalized CCSD obtained with microphone separation vector $r = 5.5$ cm, $\theta = 1.33$ rad, $\phi = 2.07$ rad. ●, prediction of CCSD given by least-squares fit procedure.

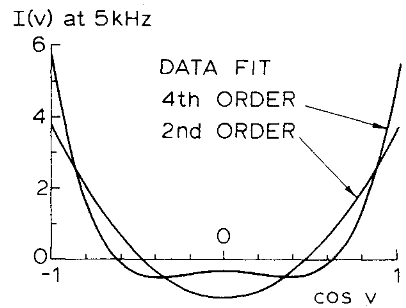

Fig. 4 Plane-wave weighting function models obtained using least-squares fit procedure, frequency 5 kHz.

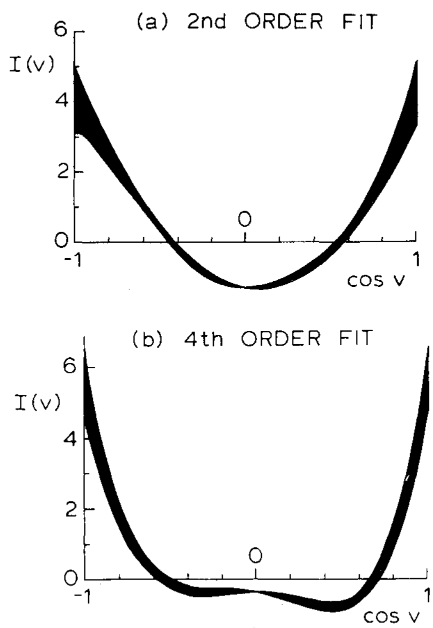

Fig. 5 Plane-wave weighting function models, frequency 5 kHz. Two families of models with comparable harmonic coefficients and statistical properties. Band indicates spread within each family.

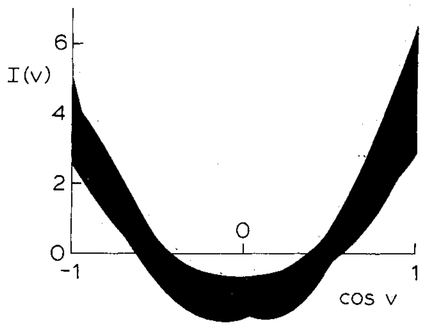

Fig. 6 Plane-wave weighting function estimates. Models from frequency range 2-5 kHz fall within the band shown.

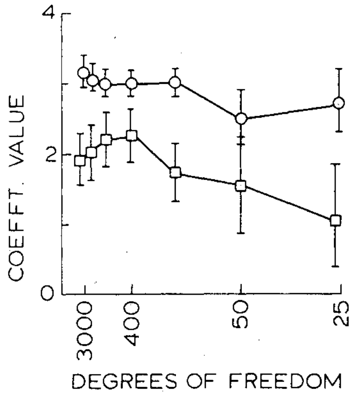

Fig. 7 Stability of coefficients in a two-harmonic model to statistical errors in data. □, value of coefficient I_2^0; ○ value of coefficient I_4^0. Bars denote standard error of each coefficient. Degrees of freedom axis scale is $n^{-1/2}$, which is proportional to the rms statistical error in the estimate of CCSD.

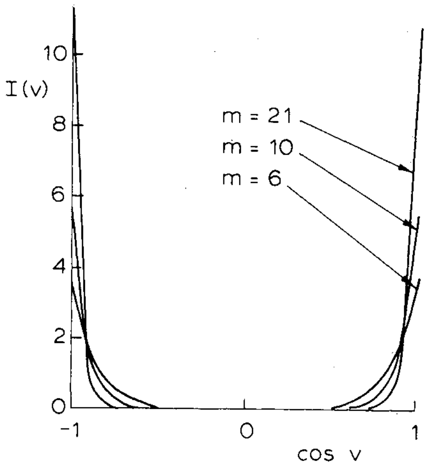

Fig. 8 Theoretical plane-wave weighting functions. Functions proportional to $|\cos^m v|$ with $m=6, 10, 21$.

test duct may be modeled by a function of the form

$$I(\omega,v) \propto |\cos v|^m \tag{7}$$

with $m \simeq 10$. The models obtained from the current experiment were too low order to demonstrate convergence, and a range of m from 6 to 21 (illustrated in Fig. 8) is possible. However, the models are sufficient to suggest that the in-duct modal power distribution is far from uniform.

Most of the sound power, in fact, is concentrated in modes with large axial wave numbers. The observed distribution may be a consequence of the duct inlet geometry. This hypothesis is supported by theoretical studies of the excitation of a duct by a tapered inlet, leading from a reverberation chamber containing a diffuse sound field.

Modes can be observed propagating in both directions at all frequencies. This indicates that the NTF acts as a coupled system, with modes in the duct being driven by the diffuse sound fields in both upstream and downstream reverberation chambers.

Although a larger scale experiment would be needed for confident modal power estimates, these results may be of value in the future practical application of the NTF.

Conclusions

The objective of the project outlined herein was to develop and apply a new data analysis technique for investigating in-duct modal sound power distributions at frequencies high enough for many duct modes to propagate.

The technique has been validated by applying it successfully to synthetic data. It has been applied to data from a microphone traverse experiment in a duct. With 12 traverse positions, information has been obtained about the structure of a duct field at frequencies high enough to allow as many as 500 modes to propagate. The new technique thus may form the basis of an economical way of investigating many duct acoustic systems.

Appendix: Relation Between Transmitted Power per Mode and the Plane-Wave Weighting Function $I(\omega,u,v)$

The definition of the polar angle v in Eq. (3) implies that the mode eigenvalue $\bar{k}$ is related to v by

$$\bar{k} = (k^2 - k_{ax}^2)^{1/2} = k\sin v \tag{A1}$$

The following derivation uses Eq. (A1), together with the asymptotic expression

$$N \simeq \bar{k}^2 S/2\pi \qquad (N \gg 1) \tag{A2}$$

for the number of modes having eigenvalues less than $\bar{k}$, to relate the acoustic power per mode at high frequencies to the weighting function $I(\omega,u,v)$.

Consider the modes corresponding to the polar angle range $(v, v+dv)$, at a given frequency. The eigenvalue range $(\bar{k}, \bar{k}+d\bar{k})$ follows from Eq. (A1). In particular,

$$d\bar{k} = k\cos v \; dv$$

and thus the number of such modes follows from Eq. (A2) as

$$dN = (\bar{k}S/\pi)k\cos v \; dv = (k^2 S/\pi)\sin v \cos v \; dv \tag{A3}$$

But according to the free-wave interpretation of the sound field, the power carried down the duct by this group of modes is proportional to

$$dW = \cos v(2\pi \sin v dv) I(\omega,v) \quad \text{(axial symmetry assumed)‡} \tag{A4}$$

‡The restriction to axial symmetry in Eq. (A4) [implying that $I(\omega,v,u)$ is independent of the azimuthal angle u] could be removed by replacing I with its azimuthally averaged value. The free-wave nature of the field has not been established for nonaxisymmetric fields, however, except in the special case of a rectangular hard-walled duct. Therefore, it remains a matter of conjecture whether such fields can be described by a plane-wave weighting function.

In Eq. (A4), the factor $I(\omega,v)$ represents the normalized mean square pressure (per unit solid angle) associated with the range of propagation directions $(v,v+dv)$. $2\pi \sin v dv$ is the solid angle subtended by this range of directions. Finally, the $\cos v$ factor gives the axial component of the associated intensity.

Comparison of Eqs. (A3) and (A4) shows that the power per mode transmitted down the duct, which is proportional to dW/dN, is given at any frequency (apart from a constant factor) by the plane-wave weighting function $I(\omega,v)$. This is the required result.

Acknowledgments

Useful guidance on the spherical harmonic fitting procedure was provided at the beginning of the investigation by Dr. D.R. Barraclough, Geomagnetism Unit, IGS Edinburgh. The first author acknowledges the joint financial support of the Science and Engineering Research Council of the United Kingdom and the National Gas Turbine Establishment through a CASE award.

References

[1] Baxter, S.M. and Morfey, C.L., "Modal Power Distribution in Ducts at High Frequencies," *AIAA Journal,* Vol. 21, Jan 1983, pp. 74-80.

[2] Baxter, S.M., "Acoustic Modal Power Distribution in a Duct at High Frequencies," Institute of Sound and Vibration Research, Southampton University, Southampton, England, Tech. Rept. 111, 1981.

[3] Baxter, S.M., "Modal Power Distribution in a Duct at High Frequency," Ph.D. Thesis, Institute of Sound and Vibration Research, Southampton University, Southampton, England, 1983.